This Technical Supplement is associated with the article “Portfolio Frontiers with Restrictions to Tracking Error Volatility and Value at Risk”, submitted to the Journal of Banking and Finance. The paper introduces a new portfolio frontier, the Fixed VaR-TEV Frontier (FVTF). In doing so, the more general setup is presented in section 4. Specifically, a scenario analysis is conducted under the following assumptions:

1. $\Delta_1 > 0$: the horizontal axis of the CTF has a positive slope in $(\sigma^2_P, \mu_P)$ space,
2. $z_\theta > \sqrt{d}$: the confidence level of the managers is high,
3. $T_0 < T_H$: the CTF and the MVF do not intersect.

The scope of this Technical Supplement is to discuss all the other scenarios regarding the interactions between portfolio frontiers when restrictions upon TEV and VaR are jointly imposed.

1 Horizontal axis of the ellipse with positive slope, high confidence level

1.1 One contact between the MVF and the CTF

In this scenario $\Delta_1 > 0$ and $z_\theta > \sqrt{d}$ are kept, while $T_0 = T_H$ is imposed to determine a unique intersection between the MVF and the CTF. The role of tangency portfolio $H = (\sigma^2_H + \Delta_1^2/d, \mu_B)$ is crucial in this analysis because it might also occur that the CVF is tangent to the MVF in $H$: this is a special case in which the FVTF is given by portfolio $M \equiv H \equiv K$, the minimum bound in Figure 3 (b) and the medium bound in Figure 3 (d) become the same VaR restriction and the strong bound in Figure 3 (c) cannot be imposed because $V_M = V_K$. When $M \equiv H \equiv K$, the slope of CVF is

$$z_H^H = \sqrt{d + \frac{d^2 \sigma^2_C}{\Delta_1^2}},$$

(T-1)
where \( z^H_\theta > \sqrt{d} \) by definition and
\[
z^*_\theta > z^H_\theta.
\] (T-2)

**Proof of equation (T-1)**
Given \( M \equiv (\sigma_1^2 + d\sigma_1^2/(z^2_\theta - d), \mu_C + d\sigma_C/\sqrt{z^2_\theta - d}) \) and \( H \equiv (\sigma_1^2 + \Delta_1^2/d, \mu_B) \), if \( M \equiv H \), it follows that
\[
\mu_C + \frac{d\sigma_C}{\sqrt{z^2_\theta - d}} = \mu_B \quad \Rightarrow \quad \sqrt{z^2_\theta - d} = \frac{d\sigma_C}{\Delta_1}.
\]
Given that \( d > 0, \sigma_C > 0, \Delta_1 > 0 \) and \( z^2_\theta - d > 0 \), by definition, the solution is
\[
z^H_\theta = \sqrt{d} + \frac{d^2\sigma_C}{\Delta_1}.
\]
Moreover, the relationship \( z^H_\theta > \sqrt{d} \) is straightforward:
\[
z^H_\theta = \sqrt{d} + \frac{\Delta_1^2 - d\sigma_C^2}{\Delta_1^2}.
\]

**Proof of equation (T-2)**
From equations (19) and (T-1) it follows that
\[
\frac{d}{2\Delta_1}(\sigma_1 + \sigma_2) > \sqrt{d + \frac{d^2\sigma_C^2}{\Delta_1^2}}
\]
therefore
\[
\frac{\sigma_1 + \sigma_2}{2} > \sqrt{\sigma_1^2 + \frac{\Delta_1^2}{d}} = \sigma_H.
\]
Given that portfolio \( H \) lies on the Mean-Variance Frontier, it surely has a lower risk than the average of risks in portfolios \( J_1 \) and \( J_2 \), and this completes the proof.

All the other scenarios with \( V_0 > V_K = V_H = V_M \) remain identical to those illustrated in Figure 3. Furthermore, when \( z_\theta = z^H_\theta \), equation (T-2) indicates that \( \hat{V} = V_2 \).

When \( z_\theta \neq z^H_\theta \), and therefore \( M \neq H \neq K \), two different scenarios could arise: if \( \sqrt{d} < z_\theta < z^H_\theta \), it follows that \( \mu_B < \mu_K < \mu_M \) while, if \( z_\theta > z^H_\theta \), it follows that \( \mu_M < \mu_K < \mu_B \), as shown in Figure T-1. In both cases, \( V_M < V_K < V_H \) and minimum, strong and medium bounds exist.

### 1.2 Two contacts between the MVF and the CTF

When \( T_0 > T_H \), the TEV constraint is feeble and the CTF intersects the MVF in two distinct portfolios, thus forming the arc \( H_1H_2 \) whose length augments when \( \Psi > 0 \) in equation (5) increases (see Palomba, 2008); in this context, portfolio \( H \in H_1H_2 \) by definition, \( \mu_{H_2} < \mu_B < \mu_{H_1} \), and the FVTF is the same as defined in the previous sections. However, depending on \( z_\theta \),...
From the analytical perspective, when a low confidence level \((z_\theta \leq \sqrt{d})\) applies, the CVF cannot be tangent to the two hyperbolic frontiers MVF and MTF in \((\sigma_P, \mu_P)\) space. The whole analysis is summarised by Figure T-2, in which the condition \(T_0 < T_H\) is adopted for simplicity.
(a) **strong bound:** as clearly shown in Alexander & Baptista (2008), an intersection always exists between the straight line CVF and the frontiers MVF and MTF (portfolios $M$ and $R$). When $V_0 < V_K$, asset managers have to make a choice between VaR and TEV because it is impossible to obtain $V_0$ and $T_0$ at the same time.

(b) **medium bound:** in this case $V_0 = V_K$ and the FVTF is given by $K$, which is the tangency portfolio between the CVF and the CTF: portfolio $K$ represents the sole position at which manager can satisfy both VaR and TEV restrictions.

(c) **intermediate bound:** when $V_K < V_0 < V_1$ the CVF intersects the MTF outside the CTF, thus the FVTF is composed of $K_1K_2$ and $K_1K_2$, where $K_1$ and $K_2$ are the contact portfolios belonging to both the CVF and the ellipse.

(d) **maximum bound:** “maximum” because it corresponds to the more stringent VaR restriction at which the FVTF has a portfolio in common with MTF: specifically, the bound $V_0 = V_1$ implies that the CVF passes through portfolio $R = J_1$, thus FVTF is simply provided by the segment $K_2J_1$ and arc $K_2J_1$.

(e) **large bound:** in such a situation $V_1 < V_0 < V_2$, where $V_2$ is defined as the VaR restriction in portfolio $J_2$: the FVTF is generally composed by arcs $K_2J_1$ and $RJ_1$ and segment $K_2R$ that belongs to he straight line CVF. Portfolio $R$ is the intersection between the MTF and the CVF.

(f) **larger bound:** when $V_0 = V_2$, the straight line CVF passes through portfolio $J_2$ and the portfolios composing the FVTF corresponds to arcs $J_1J_2$ belonging to both the MTF and the CTF (to the left of MTF).

(g) **no bound:** when $V_0 > V_2$, the VaR constraint is uneffective and the FVTF is as described in the larger bound scenario.

When $T_0 \geq T_H$, all the above scenarios remain substantially unaltered and the analysis could therefore be extended to situations in which the MVF and the CTF intersect.

---

1The slope $z_θ = \sqrt{d}$ represents the only exception: Alexander & Baptista (2008) show that when $V_0 \leq -\mu_C$, the CVF does not intersect the MVF. Moreover, when $-\mu_C < V_0 \leq -\mu_C + \sqrt{d\sigma_H}$, the CVF only intersects the MVF: in this case, the contact portfolio $R$ does not exist.
Figure T-2: $\Delta_1 > 0$, low confidence level, $z_\theta < \sqrt{\Delta}$, $T_0 < T_H$.
3 Horizontal axis of the ellipse with non positive slope

When $\Delta_1 < 0$, the horizontal axis of the ellipse CTF has a negative slope in $(\sigma_P^2, \mu_P)$ space, while it has zero slope when $\mu_B = \mu_C$. Under these assumptions, the scenarios plotted in Figures 3 of the paper and T-2 are substantially confirmed as are the discussions of the previous sections. In such a situation, the relevant differences are:

(i) $\sigma_1 \leq \sigma_2$ and $\mu_1 > \mu_2$, thus no feasible VaR constraints pass through $J_1 \equiv (\sigma_1, \mu_1)$ and $J_2 \equiv (\sigma_2, \mu_2)$: in particular, the slope $z_0^H$ in equation (19) would be negative when $\mu_B < \mu_C$ or infinite when the ellipse in the $(\sigma_P^2, \mu_P)$ space has a horizontal axis;

(ii) the relationship $V_1 < V_2$ applies for any $0.5 < \theta < 1$;

(iii) scenarios similar to those documented in Figure T-1 are not available. Portfolio $H$ lies on the inefficient arc of the MVF, thus it can not coincide with the tangency portfolio $M$.

4 An empirical example

This section presents the same empirical analysis that has been conducted in section 5 of the paper. All the results are shown in Table T-1. Here, the principal remarks are:

- the DJ Eurostoxx 50 index is the benchmark portfolio,
- $\Delta_1 < 0$,
- the above condition determines the slopes $z_0^H$ and $z_0^H$ cannot be calculated,
- $\mu_R \notin [\mu_2, \mu_1]$, rendering the benchmark extreme ($T_R = 80.674$).
Table T-1: Empirical results (Benchmark portfolio: DJ Eurostoxx 50 Index)

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Expected Return</th>
<th>TEV constraint ($T_0$): 20.000</th>
<th>$\Delta_1$: -0.365, $\Delta_2$: 57.383</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tangency TEV ($T_H$): 57.328, $\Psi$: -87.520</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>The benchmark is extreme ($T_0 &lt; T_R$)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Intermediate Bound:**
VaR constraint ($V_0$): 15.000
High confidence level, $\theta$: 99%, $z_\theta$: 2.326, Threshold ($\sqrt{\theta}$): 1.531
$\hat{V} = V_2$: 31.575
Intersections in $(\sigma_P, \mu_P)$ space: $M_1 \equiv (18.231, 27.411)$ and $M_2 \equiv (6.570, 0.285)$
Intersections in $(\sigma_P, \mu_P)$ space: $K_1 \equiv (9.585, 7.299)$ and $K_2 \equiv (7.256, 1.880)$
Efficiency loss: $\delta_{K_1}$: 34.098, $\delta_{K_2}$: 9.846

<table>
<thead>
<tr>
<th>Portfolios:</th>
<th>$P$</th>
<th>$T$</th>
<th>$J$</th>
<th>$AB$</th>
<th>$B$</th>
<th>$C$</th>
<th>$Q$</th>
<th>$J_1$</th>
<th>$J_2$</th>
<th>$H$</th>
<th>$M$</th>
<th>$K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exp. Return</td>
<td>5.000</td>
<td>5.000</td>
<td>5.000</td>
<td>5.000</td>
<td>0.985</td>
<td>1.350</td>
<td>75.494</td>
<td>7.833</td>
<td>-5.863</td>
<td>0.985</td>
<td>10.097</td>
<td>5.012</td>
</tr>
<tr>
<td>Variance</td>
<td>48.369</td>
<td>105.700</td>
<td>63.961</td>
<td>73.911</td>
<td>100.070</td>
<td>42.687</td>
<td>2387.3</td>
<td>117.940</td>
<td>122.191</td>
<td>42.743</td>
<td>75.317</td>
<td>64.043</td>
</tr>
<tr>
<td>Sharpe Ratio</td>
<td>0.719</td>
<td>0.486</td>
<td>0.625</td>
<td>0.582</td>
<td>0.098</td>
<td>0.207</td>
<td>1.545</td>
<td>0.721</td>
<td>-0.530</td>
<td>0.151</td>
<td>1.163</td>
<td>0.626</td>
</tr>
<tr>
<td>TEV</td>
<td>64.203</td>
<td>6.874</td>
<td>20.000</td>
<td>166.280</td>
<td>-</td>
<td>57.385</td>
<td>2425.1</td>
<td>20.000</td>
<td>20.000</td>
<td>57.328</td>
<td>92.736</td>
<td>20.000</td>
</tr>
<tr>
<td>Information Ratio</td>
<td>0.063</td>
<td>0.584</td>
<td>0.201</td>
<td>0.024</td>
<td>-</td>
<td>0.0064</td>
<td>0.0307</td>
<td>0.342</td>
<td>-0.342</td>
<td>-0.098</td>
<td>0.201</td>
<td>0.170</td>
</tr>
</tbody>
</table>

**Large Bound:**
VaR constraint ($V_0$): 15.00, with $V_0 > \mu_C$
Low confidence level, $\theta$: 0.937
Threshold $z_\theta = \sqrt{\theta}$: 1.531
$V_1$: 8.793, $V_2$: 22.786
$\mu_P > \mu_R \Rightarrow AB = T$

<table>
<thead>
<tr>
<th>Portfolios:</th>
<th>$AB$</th>
<th>$M$</th>
<th>$K_2$</th>
<th>$K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exp. Return</td>
<td>5.000</td>
<td>-3.764</td>
<td>-2.723</td>
<td>6.193</td>
</tr>
<tr>
<td>Variance</td>
<td>105.700</td>
<td>53.842</td>
<td>64.289</td>
<td>74.477</td>
</tr>
<tr>
<td>Risk</td>
<td>10.281</td>
<td>7.338</td>
<td>8.018</td>
<td>8.630</td>
</tr>
<tr>
<td>Sharpe Ratio</td>
<td>0.486</td>
<td>-0.513</td>
<td>-0.340</td>
<td>0.718</td>
</tr>
<tr>
<td>Alpha</td>
<td>4.015</td>
<td>-4.750</td>
<td>-3.708</td>
<td>5.208</td>
</tr>
<tr>
<td>TEV</td>
<td>6.874</td>
<td>66.949</td>
<td>20.000</td>
<td>20.000</td>
</tr>
<tr>
<td>Information Ratio</td>
<td>0.584</td>
<td>-0.071</td>
<td>-0.185</td>
<td>0.260</td>
</tr>
<tr>
<td>Efficiency Loss</td>
<td>57.328</td>
<td>-</td>
<td>14.592</td>
<td>21.786</td>
</tr>
<tr>
<td>VaR</td>
<td>10.742</td>
<td>15.000</td>
<td>15.000</td>
<td>7.021</td>
</tr>
</tbody>
</table>
References
