Portfolio Frontiers with Restrictions to Tracking Error Volatility and Value at Risk

Giulio Palomba Luca Riccetti

Technical Supplement

This Technical Supplement is associated with the article "Portfolio Frontiers with Restrictions to Tracking Error Volatility and Value at Risk", submitted to the Journal of Banking and Finance. The paper introduces a new portfolio frontier, the Fixed VaR-TEV Frontier (FVTF). In doing so, the more general setup is presented in section 4. Specifically, a scenario analysis is conducted under the following assumptions:

1. $\Delta_1 > 0$: the horizontal axis of the CTF has a positive slope in (σ_P^2, μ_P) space,
2. $z_\theta > \sqrt{d}$: the confidence level of the managers is high,
3. $T_0 < T_H$: the CTF and the MVF do not intersect.

The scope of this Technical Supplement is to discuss all the other scenarios regarding the interactions between portfolio frontiers when restrictions upon TEV and VaR are jointly imposed.

1 Horizontal axis of the ellipse with positive slope, high confidence level

1.1 One contact between the MVF and the CTF

In this scenario $\Delta_1 > 0$ and $z_\theta > \sqrt{d}$ are kept, while $T_0 = T_H$ is imposed to determine a unique intersection between the MVF and the CTF. The role of tangency portfolio $H \equiv (\sigma_C^2 + \Delta_1^2/d, \mu_B)$ is crucial in this analysis because it might also occur that the CVF is tangent to the MVF in H: this is a special case in which the FVTF is given by portfolio $M \equiv H \equiv K$, the minimum bound in Figure 3 (b) and the medium bound in Figure 3 (d) become the same VaR restriction and the strong bound in Figure 3 (c) cannot be imposed because $V_M = V_K$. When $M \equiv H \equiv K$, the slope of CVF is

$$z_\theta^H = \sqrt{d + \frac{d^2\sigma_C^2}{\Delta_1^2}}, \tag{T-1}$$

where $z_\theta^H > \sqrt{d}$ by definition and

$$z_\theta^* > z_\theta^H. \tag{T-2}$$

Proof of equation (T-1)

Given $M \equiv (\sigma_C^2 + d\sigma_C^2/(z_\theta^2 - d), \mu_C + d\sigma_C/\sqrt{z_\theta^2 - d})$ and $H \equiv (\sigma_C^2 + \Delta_1^2/d, \mu_B)$, if $M \equiv H$, it follows that

$$\mu_C + \frac{d\sigma_C}{\sqrt{z_\theta^2 - d}} = \mu_B \qquad \Rightarrow \qquad \sqrt{z_\theta^2 - d} = \frac{d\sigma_C}{\Delta_1}.$$

Given that $d > 0$, $\sigma_C > 0$, $\Delta_1 > 0$ and $z_\theta^2 - d > 0$, by definition, the solution is

$$z_\theta^H = \sqrt{d + \frac{d^2\sigma_C^2}{\Delta_1^2}}.$$

Moreover, the relationship $z_\theta^H > \sqrt{d}$ is straighforward:

$$z_\theta^H = \sqrt{d}\sqrt{\frac{\Delta_1^2 - d\sigma_C^2}{\Delta_1^2}}$$

Proof of equation (T-2)

From equations (19) and (T-1) it follows that

$$\frac{d}{2\Delta_1}(\sigma_1 + \sigma_2) > \sqrt{d + \frac{d^2\sigma_C^2}{\Delta_1^2}}$$

therefore

$$\frac{\sigma_1 + \sigma_2}{2} > \sqrt{\sigma_C^2 + \frac{\Delta_1^2}{d}} = \sigma_H.$$

Given that portfolio H lies on the Mean-Variance Frontier, it surely has a lower risk than the average of risks in portfolios J_1 and J_2, and this completes the proof.

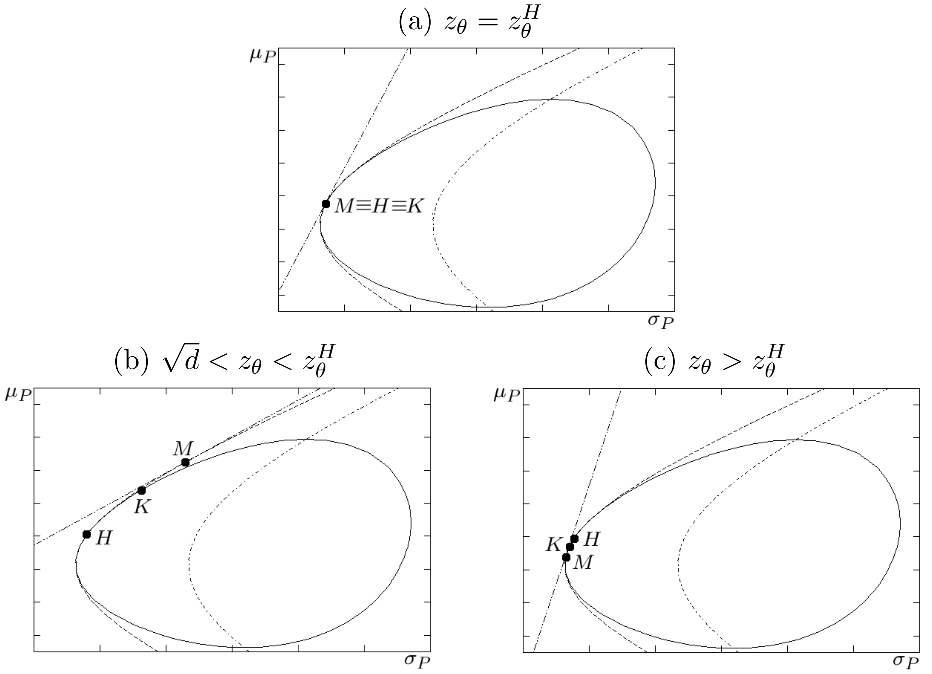
All the other scenarios with $V_0 > V_K = V_H = V_M$ remain identical to those illustrated in Figure 3. Furthermore, when $z_\theta = z_\theta^H$, equation (T-2) indicates that $\hat{V} = V_2$.

When $z_\theta \neq z_\theta^H$, and therefore $M \neq H \neq K$, two different scenarios could arise: if $\sqrt{d} < z_\theta < z_\theta^H$, it follows that $\mu_B < \mu_K < \mu_M$ while, if $z_\theta > z_\theta^H$, it follows that $\mu_M < \mu_K < \mu_B$, as shown in Figure T-1. In both cases, $V_M < V_K < V_H$ and minimum, strong and medium bounds exist.

1.2 Two contacts between the MVF and the CTF

When $T_0 > T_H$, the TEV constraint is feeble and the CTF intersects the MVF in two distinct portfolios, thus forming the arc $\widehat{H_1H_2}$ whose length augments when $\Psi > 0$ in equation (5) increases (see Palomba, 2008); in this context, portfolio $H \in \widehat{H_1H_2}$ by definition, $\mu_{H_2} < \mu_B < \mu_{H_1}$ and the FVTF is the same as defined in the previous sections. However, depending on z_θ,

Figure T-1: $\Delta_1 > 0$, high confidence level, $T_0 = T_H$. All Figures are plotted with the CVF passing through portfolio M



Ψ and V_0, each of the following relationships may occur: $\widehat{K_1K_2} \cap \widehat{H_1H_2} = \varnothing$, $\widehat{K_1K_2} \cap \widehat{H_1H_2} \neq \varnothing$, $\widehat{K_1K_2} \subset \widehat{H_1H_2}$ and $\widehat{H_1H_2} \subset \widehat{K_1K_2}$.

In practical situations, an interesting scenario emerges when the condition $M \in \widehat{H_1H_2}$ holds: in such a situation, the minimum VaR bound $V_0 = V_M$ is sufficient for obtaining a portfolio which satisfies both TEV and VaR restrictions. Conversely, when $M \notin \widehat{H_1H_2}$, the expected return of the tangency portfolio M could be greater than that of portfolio H_1 or less than that of portfolio H_2: in the former case, M lies on the MVF efficient set, to the right of H_1, where the tangency can only be reached for slopes z_θ that are close to the MVF asymptotic slope $\sqrt{d}$. In the latter case, the tangency may only occur when $\Psi > 0$ is sufficiently small to guarantee the condition $\mu_C < \mu_M < \mu_{H_2}$.

2 Low confidence level

From the analytical perspective, when a low confidence level ($z_\theta \leq \sqrt{d}$) applies, the CVF cannot be tangent to the two hyperbolic frontiers MVF and MTF in (σ_P, μ_P) space. The whole analysis is summarised by Figure T-2, in which the condition $T_0 < T_H$ is adopted for simplicity.

(a) strong bound: as clearly shown in Alexander & Baptista (2008), an intersection always exists between the straight line CVF and the frontiers MVF and MTF (portfolios M and R).[1] When $V_0 < V_K$, asset managers have to make a choice between VaR and TEV because it is impossible to obtain V_0 and T_0 at the same time.

(b) medium bound: in this case $V_0 = V_K$ and the FVTF is given by K, which is the tangency portfolio between the CVF and the CTF: portfolio K represents the sole position at which manager can satisfy both VaR and TEV restrictions.

(c) intermediate bound: when $V_K < V_0 < V_1$ the CVF intersects the MTF outside the CTF, thus the FVTF is composed of $\overline{K_1K_2}$ and $\widehat{K_1K_2}$, where K_1 and K_2 are the contact portfolios belonging to both the CVF and the ellipse.

(d) maximum bound: "maximum" because it corresponds to the more stringent VaR restriction at which the FVTF has a portfolio in common with MTF: specifically, the bound $V_0 = V_1$ implies that the CVF passes through portfolio $R \equiv J_1$, thus FVTF is simply provided by the segment $\overline{K_2J_1}$ and arc $\widehat{K_2J_1}$.

(e) large bound: in such a situation $V_1 < V_0 < V_2$, where V_2 is defined as the VaR restriction in portfolio J_2; the FVTF is generally composed by arcs $\widehat{K_2J_1}$ and $\widehat{RJ_1}$ and segment $\overline{K_2R}$ that belongs to he straight line CVF. Portfolio R is the intersection between the MTF and the CVF.

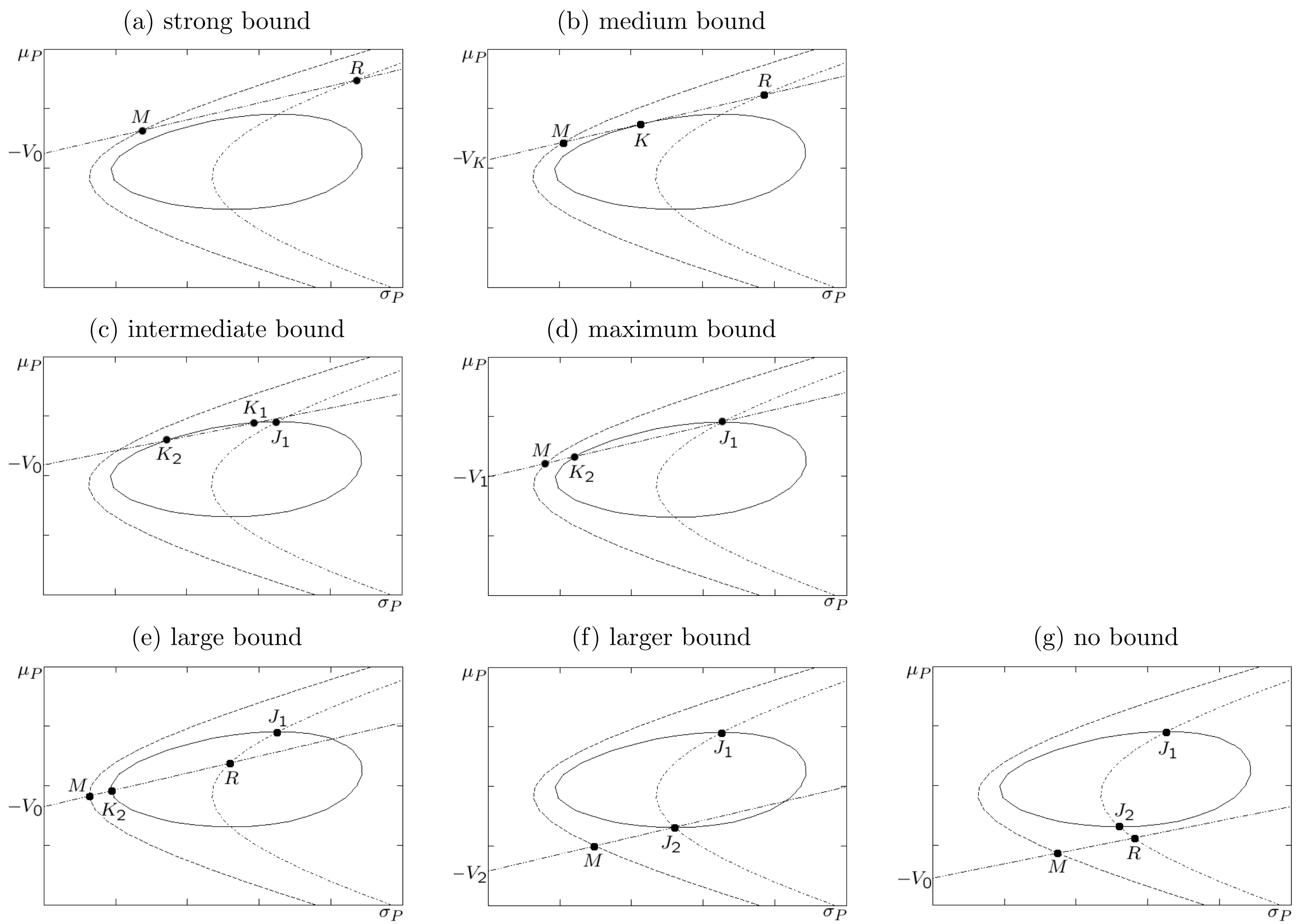
(f) larger bound: when $V_0 = V_2$, the straight line CVF passes through portfolio J_2 and the portfolios composing the FVTF corresponds to arcs $\widehat{J_1J_2}$ belonging to both the MTF and the CTF (to the left of MTF).

(g) no bound: when $V_0 > V_2$, the VaR constraint is uneffective and the FVTF is as described in the larger bound scenario.

When $T_0 \geq T_H$, all the above scenarios remain substantially unaltered and the analysis could therefore be extended to situations in which the MVF and the CTF intersect.

[1] The slope $z_\theta = \sqrt{d}$ represents the only exception: Alexander & Baptista (2008) show that when $V_0 \leq -\mu_C$, the CVF does not intersect the MVF. Moreover, when $-\mu_C < V_0 \leq -\mu_C + \sqrt{d\delta_B}$, the CVF only intersects the MVF: in this case, the contact portfolio R does not exist.

Figure T-2: $\Delta_1 > 0$, low confidence level, $z_\theta < \sqrt{d}$, $T_0 < T_H$



3 Horizontal axis of the ellipse with non positive slope

When $\Delta_1 < 0$, the horizontal axis of the ellipse CTF has a negative slope in (σ_P^2, μ_P) space, while it has zero slope when $\mu_B = \mu_C$. Under these assumptions, the scenarios plotted in Figures 3 of the paper and T-2 are substantially confirmed as are the discussions of the previous sections. In such a situation, the relevant differences are:

(i) $\sigma_1 \leq \sigma_2$ and $\mu_1 > \mu_2$, thus no feasible VaR constraints pass through $J_1 \equiv (\sigma_1, \mu_1)$ and $J_2 \equiv (\sigma_2, \mu_2)$: in particular, the slope z_θ^* in equation (19) would be negative when $\mu_B < \mu_C$ or infinite when the ellipse in the (σ_P^2, μ_P) space has a horizontal axis;

(ii) the relationship $V_1 < V_2$ applies for any $0.5 < \theta < 1$;

(iii) scenarios similar to those documented in Figure T-1 are not available. Portfolio H lies on the inefficient arc of the MVF, thus it can not coincide with the tangency portfolio M.

4 An empirical example

This section presents the same empirical analysis that has been conducted in section 5 of the paper. All the results are shown in Table T-1. Here, the principal remarks are:

- the DJ Eurostoxx 50 index is the benchmark portfolio,
- $\Delta_1 < 0$,
- the above condition determines the slopes z_θ^* and z_θ^H cannot be calculated,
- $\mu_R \notin [\mu_2, \mu_1]$, rendering the benchmark extreme ($T_R = 80.674$).

Table T-1: Empirical results (Benchmark portfolio: DJ Eurostoxx 50 Index)

Portfolio Expected Return: 5.000
TEV constraint (T_0): 20.000
Δ_1: -0.365, Δ_2: 57.383
Tangency TEV (T_H): 57.328, Ψ: -87.520
The benchmark is extreme ($T_0 < T_R$)

Intermediate Bound:
VaR constraint (V_0): 15.000
High confidence level, θ: 99%, z_θ: 2.326, Threshold ($\sqrt{d}$): 1.531
$\hat{V} = V_2$: 31.575
Intersections in (σ_P, μ_P) space: $M_1 \equiv (18.231, 27.411)$ and $M_2 \equiv (6.570, 0.285)$
Intersections in (σ_P, μ_P) space: $K_1 \equiv (9.585, 7.299)$ and $K_2 \equiv (7.256, 1.880)$
Efficiency loss: δ_{K_1}: 34.098, δ_{K_2}: 9.846

Portfolios:	P	T	J	AB	B	C	Q	J_1	J_2	H	M	K	R
Exp. Return	5.000	5.000	5.000	5.000	0.985	1.350	75.494	7.833	-5.863	0.985	10.097	5.012	14.739
Variance	48.369	105.700	63.961	73.911	100.070	42.687	2387.3	117.940	122.191	42.743	75.317	64.043	176.470
Risk	6.955	10.281	7.997	8.597	10.004	6.533	48.860	10.860	11.054	6.538	8.679	8.003	13.284
Sharpe Ratio	0.719	0.486	0.625	0.582	0.098	0.207	1.545	0.721	-0.530	0.151	1.163	0.626	1.109
Alpha	4.015	4.015	4.015	4.015	-	0.365	74.509	6.848	-6.848	-	9.111	4.027	13.753
TEV	64.203	6.874	20.000	166.280	-	57.385	2425.1	20.000	20.000	57.328	92.736	20.000	80.674
Information Ratio	0.063	0.584	0.201	0.024	-	0.0064	0.0307	0.342	-0.342	-	0.098	0.201	0.170
Efficiency Loss	-	57.328	15.592	25.542	57.328	-	-	57.328	57.328	-	-	15.636	57.328
VaR	11.179	18.917	13.605	15.000	22.287	13.849	38.172	17.431	31.579	14.224	10.093	13.065	16.165

Large Bound:

VaR constraint (V_0): 15.00, with $V_0 > \mu_C$
Low confidence level, θ: 0.937
Threshold $z_\theta = \sqrt{d}$: 1.531
V_1: 8.793, V_2: 22.786
$\mu_P > \mu_R \Rightarrow AB = T$

Portfolios:	AB	M	K_2	K	R
Exp. Return	5.000	-3.764	-2.723	6.193	0.346
Variance	105.700	53.842	64.289	74.477	100.44
Risk	10.281	7.338	8.018	8.630	10.022
Sharpe Ratio	0.486	-0.513	-0.340	0.718	0.034
Alpha	4.015	-4.750	-3.708	5.208	0.639
TEV	6.874	66.949	20.000	20.000	0.174
Information Ratio	0.584	-0.071	-0.185	0.260	0.367
Efficiency Loss	57.328	-	14.529	21.786	57.328
VaR	10.742	15.000	15.000	7.021	15.000

References

Alexander, G.J. & Baptista, A.M. (2008), Active Portfolio Management with Benchmarking: Adding a Value-at-Risk Constraint. Journal of Economic Dynamics & Control 32, 779–820.

Palomba, G. (2008), Multivariate GARCH Models and Black-Litterman Approach for Tracking Error Constrained Portfolios: An Empirical Analysis. Global Business and Economics Review 10(4), 379–413.